Exercise 62

Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = x - 2\tan^{-1} x, \quad [0, 4]$$

Solution

Take the derivative of the function.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x - 2\tan^{-1} x) \\ &= 1 - 2\left(\frac{1}{1+x^2}\right) \\ &= 1 - \frac{2}{1+x^2} \\ &= \frac{(1+x^2) - 2}{1+x^2} \\ &= \frac{x^2 - 1}{x^2 + 1} \end{aligned}$$

Set what's in the numerator equal to zero, and set what's in the denominator equal to zero. Solve each equation for x.

$$x^2 - 1 = 0 \qquad\qquad x^2 + 1 = 0$$

$$x = -1 \quad \text{or} \quad x = 1 \qquad\qquad x = -i \quad \text{or} \quad x = i$$

$x = 1$ is within $[0, 4]$, so evaluate f here.

$$f(1) = 1 - 2\tan^{-1} 1 = 1 - \frac{\pi}{2} \approx -0.570796 \qquad \text{(absolute minimum)}$$

Now evaluate the function at the endpoints of the interval.

$$f(0) = 0 - 2\tan^{-1} 0 = 0$$

$$f(4) = 4 - 2\tan^{-1} 4 \approx 1.34836 \qquad \text{(absolute maximum)}$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $[0, 4]$.

The graph of the function below illustrates these results.